GEOMETRIC APPROACH TO NON-CIRCULAR CONE AND SPHERE INTERSECTION CURVES

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Abstract

Using work based by Miller 1987 for the geometric approach to the intersection of a circular cone and a sphere, we show the derivation for the intersection of a non-circular cone and sphere.

Background

Miller 1987 described a geometric approach to the intersection curve for circular cones, and spheres. Miller expressed the conical surface parametrically as

$$P(s,t) = \vec{B} + s\left(\overrightarrow{\delta(t)} + \vec{w}\right) \qquad (1),$$

where

$$\overrightarrow{\delta(t)} = \tan(\alpha)\,(\cos(t)\vec{u} + \sin(t)\vec{v}) \qquad (2).$$

For each t, $\overrightarrow{\delta(t)}$ is a vector of length $\tan\alpha$ and is perpendicular to the cone axis $\vec{w}$.

The intersection curve between a cone and sphere was found by substituting *(1)* into the implicit equation for a sphere ($(\vec{P} - \vec{B}) \cdot (\vec{P} - \vec{B}) - r^2 = 0$,

$$\left(\left(\overrightarrow{B_p} - \overrightarrow{B_o}\right) + s\left(\overrightarrow{\delta(t)} + \overrightarrow{w_p}\right)\right) \cdot \left(\left(\overrightarrow{B_p} - \overrightarrow{B_o}\right) + s\left(\overrightarrow{\delta(t)} + \overrightarrow{w_p}\right)\right) - r_o^2 = 0 \qquad (3).$$

Solving for s results in a quadratic equation (Appendix A) whose coefficients are

$$\boldsymbol{a} = \left(\overrightarrow{\delta(t)} \cdot \overrightarrow{\delta(t)}\right) + 1,$$

$$\boldsymbol{b}(\boldsymbol{t}) = 2\vec{b} \cdot \left(\overrightarrow{\delta(t)} + \overrightarrow{w_p}\right), \qquad (4)$$

$$\vec{b} = \overrightarrow{B_p} - \overrightarrow{B_o},$$

$$\boldsymbol{c} = \left(\overrightarrow{B_p} - \overrightarrow{B_o}\right) \cdot \left(\overrightarrow{B_p} - \overrightarrow{B_o}\right) - r_o^2,$$

Since $\overrightarrow{\delta(t)}$ is a vector of length $\tan\alpha$, Miller 1987 replaced the squared length of $\overrightarrow{\delta(t)} \cdot \overrightarrow{\delta(t)}$ with $\tan^2\alpha$ in the term for a resulting in the same value for all t.

Non-Circular Cones

To understand the changes required for non-circular cones, let us examine how the solution works for circular cones. The three primary areas of examination are $\overrightarrow{\delta(t)}$, the resultant vector $\overrightarrow{\delta(t)} + \vec{w}$, and a geometric representation of equation (3) illustrated in figure 1. The formula for $\overrightarrow{\delta(t)}$ in (2), is the parametric equation for a circle oriented along $\vec{u}$ and $\vec{v}$ with a radius of $\tan\alpha$. $\overrightarrow{\delta(t)}$ is perpendicular to $\vec{w}$ and the resultant vector between the two yields a vector on the conical surface. The resultant vector (i.e., $\overrightarrow{\delta(t)} + \vec{w}$) is then scaled by s which is the unknown that is solved for in (3) (Figure 1). Therefore as long as $\overrightarrow{\delta(t)} + \vec{w}$ lies upon the desired non-circular conical surface, then the solution found in (3) can be applied to non-circular cones. Rewriting (2) to evaluate a vector on an ellipse rather than a circle we have

$$\overrightarrow{\delta(t)} = (\tan(\alpha)\cos(t)\vec{u}) + (\tan(\beta)\sin(t)\vec{v}) \qquad \textit{(5).}$$

We now apply $\overrightarrow{\delta(t)}$ to (3) (Appendix A) which results in the quadratic equation coefficient formulas found in *(4)*.

Circular cones yielded $\overrightarrow{\delta(t)}$ that would always be of length $\tan\alpha$ and which restricted the a coefficient to only one value for all values of t. Visually this can be described as the same angle between the resultant vector $(\overrightarrow{\delta(t)} + \vec{w})$ and $\vec{w}$ for all values of t (i.e., $\tan\alpha = \frac{\left|\overrightarrow{\delta(t)}\right|}{|\vec{w}|}$). Non-circular cones will yield varying lengths of $\overrightarrow{\delta(t)}$ from (5), thus resulting in varying angles between $(\overrightarrow{\delta(t)} + \vec{w})$ and $\vec{w}$ hence resulting in a different conical surface than circular cones.

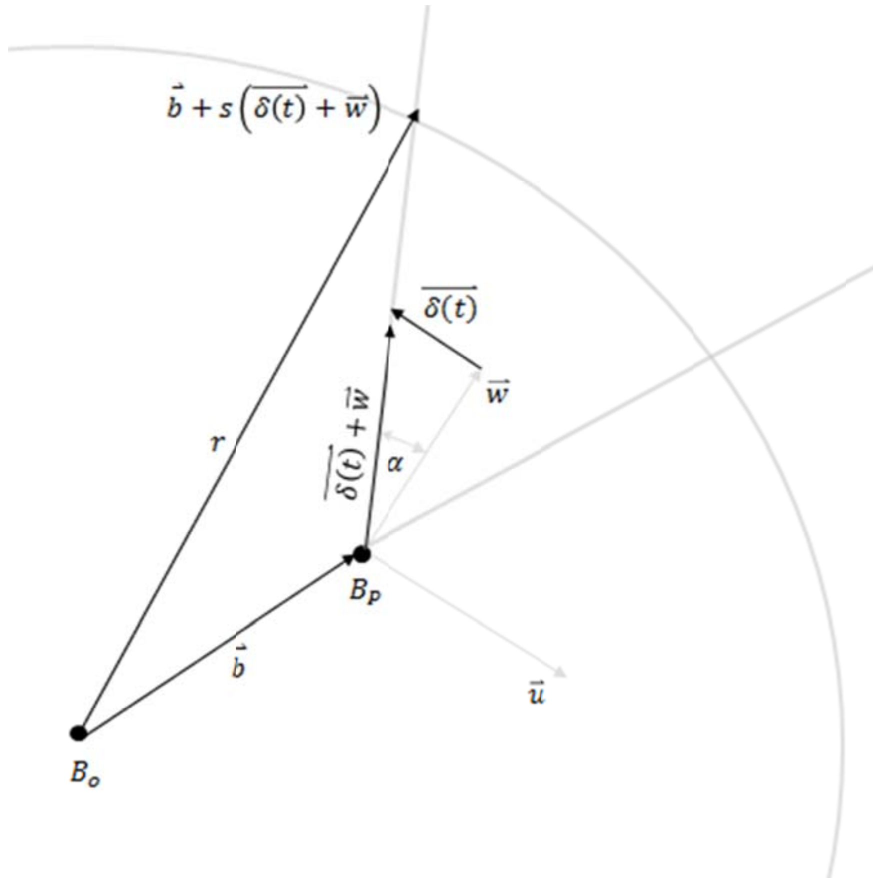

Figure 1. Cross section of a sphere centered at B_O with radius r, and cone at vertex B_P with opening angle α oriented by $\vec{u}, \vec{v}, \vec{w}$. $\vec{v}$ is directed into the diagram. The cross section geometrically illustrates the intersection curve between a sphere and cone defined by the equation, $\left(\left(\overrightarrow{B_p} - \overrightarrow{B_o}\right) + s\left(\overrightarrow{\delta(t)} + \overrightarrow{w_p}\right)\right) \cdot \left(\left(\overrightarrow{B_p} - \overrightarrow{B_o}\right) + s\left(\overrightarrow{\delta(t)} + \overrightarrow{w_p}\right)\right) - r_o^2 = 0$.

Appendix A: Derivation of the quadratic coefficients in *(3)*

Consider the following equation to find the intersection between a cone and a sphere,

$$\left(\left(\overrightarrow{B_p}-\overrightarrow{B_o}\right)+s\left(\overrightarrow{\delta(t)}+\overrightarrow{w_p}\right)\right)\cdot\left(\left(\overrightarrow{B_p}-\overrightarrow{B_o}\right)+s\left(\overrightarrow{\delta(t)}+\overrightarrow{w_p}\right)\right)-r_o^2=0.$$

Let $\overrightarrow{B_p}-\overrightarrow{B_o}$ be equal to $\vec{b}$, $\overrightarrow{\delta(t)}=\vec{v}$, and $\overrightarrow{w_p}=\vec{w}$. Rewriting the equation we get

$$\left(\vec{b}+s(\vec{v}+\vec{w})\right)\cdot\left(\vec{b}+s(\vec{v}+\vec{w})\right)-r_o^2=0.$$

Distribute the s, and we get

$$\left(\vec{b}+s\vec{v}+s\vec{w}\right)\cdot\left(\vec{b}+s\vec{v}+s\vec{w}\right)-r_o^2=0.$$

Now expand each vector's x component and multiply through to yield

$$b_x^2+\ b_xv_xs+b_xw_xs+b_xv_xs+v_x^2s^2+v_xw_xs^2+b_xw_xs+v_xw_xs^2+w_x^2s^2.$$

Group like terms

$$b_x^2+2(b_xv_xs)+2(b_xw_xs)+v_x^2s^2+2(v_xw_xs^2)+w_x^2s^2.$$

Now factor out an s^2

$$s^2(v_x^2+2(v_xw_x)+w_x^2)+s\big(2(b_xv_x)+2(b_xw_x)\big)+b_x^2.$$

The equation above has been organized to resemble a quadratic equation ($ax^2+bx+c=0$). For neatness and clarity purposes only the x components are presented, and like terms for y and z only need to be inserted into each quadratic coefficient (i.e., a, b, and c). Let's finish deriving each quadratic coefficient including the y and z values for the vectors. For the c coefficient

$$\boldsymbol{c}=\left(\boldsymbol{b_x^2}+\boldsymbol{b_y^2}+\boldsymbol{b_z^2}\right)-\boldsymbol{r^2}.$$

$b_x^2+b_y^2+b_z^2$ is the squared length of $\vec{b}$, which we had reduced from $\overrightarrow{B_p}-\overrightarrow{B_o}$. This now equals Miller's form of c defined by

$$c=\ \left(\overrightarrow{B_p}-\overrightarrow{B_o}\right)\cdot\left(\overrightarrow{B_p}-\overrightarrow{B_o}\right)-r_o^2.$$

For the b coefficient we rewrite $s\big(2(b_xv_x)+2(b_xw_x)\big)$ by factoring out $2\vec{b}$ and we get

$$\boldsymbol{b}=\ \boldsymbol{2}\begin{bmatrix}\boldsymbol{b_x}\\\boldsymbol{b_y}\\\boldsymbol{b_z}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{v_x}+\boldsymbol{w_x}\\\boldsymbol{v_y}+\boldsymbol{w_y}\\\boldsymbol{v_z}+\boldsymbol{w_z}\end{bmatrix}$$

This is now equal to Millers form of b defined by

$$b(t) = 2\vec{b} \cdot \left(\overrightarrow{\delta(t)} + \overrightarrow{w_p}\right).$$

For the a coefficient we get

$$a = \left(v_x^2 + v_y^2 + v_z^2\right) + 2\left(v_x w_x + v_y w_y + v_z w_z\right) + \left(w_x^2 + w_y^2 + w_z^2\right).$$

$w_x^2 + w_y^2 + w_z^2$ is the squared length of a normal vector and will always be equal to 1. $v_x^2 + v_y^2 + v_z^2$ is equal to the squared length of the vector $\vec{v}$. $v_x w_x + v_y w_y + v_z w_z$ is the dot product between perpendicular vectors and is 0. Thus a is

$$\boldsymbol{a} = (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}) + \mathbf{1}.$$

References

Miller, James R., 1987. Geometric Approaches to Nonplanar Quadric Surface Intersection Curves, *ACM Transactions on Graphics*, Vol. 6, No. 4, October 1987, Pages 274-307.